

MAS212 Linear Algebra I 2007-2008

Coursework 11 — Solutions

Exercise 1.

- (a) The number 0 is an eigenvalue of A if and only if $\det(A) = 0$, which, by the Invertible Matrix Theorem, is the case if and only if A is not invertible.
- (b) Suppose that A is invertible. If λ is an eigenvalue of A with corresponding eigenvector $\mathbf{v}$, then

$$A\mathbf{v} = \lambda\mathbf{v}.$$

Moreover, by (a) above, $\lambda \neq 0$. Multiplying the equation above by $\lambda^{-1}A^{-1}$ from the left we obtain

$$\lambda^{-1}A^{-1}A\mathbf{v} = \lambda^{-1}\lambda A^{-1}\mathbf{v},$$

so

$$\lambda^{-1}\mathbf{v} = A^{-1}\mathbf{v},$$

that is, λ^{-1} is an eigenvalue of A^{-1} (with corresponding eigenvector $\mathbf{v}$). Conversely, suppose that λ^{-1} is an eigenvalue of A with eigenvector $\mathbf{v}$, that is,

$$A^{-1}\mathbf{v} = \lambda^{-1}\mathbf{v}.$$

Multiplying the above equation by λA from the left obtain

$$\lambda AA^{-1}\mathbf{v} = \lambda\lambda^{-1}A\mathbf{v},$$

so

$$\lambda\mathbf{v} = A\mathbf{v},$$

that is λ is an eigenvalue of A .

- (c) If λ is an eigenvalue of A with corresponding eigenvector $\mathbf{v}$, then

$$A\mathbf{v} = \lambda\mathbf{v}.$$

Thus

$$A^2\mathbf{v} = A(A\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda(A\mathbf{v}) = \lambda\lambda\mathbf{v} = \lambda^2\mathbf{v},$$

so λ^2 is an eigenvalue of A^2 (with corresponding eigenvector $\mathbf{v}$). Similarly,

$$A^3\mathbf{v} = A(A^2\mathbf{v}) = A(\lambda^2\mathbf{v}) = \lambda^2(A\mathbf{v}) = \lambda\lambda^2\mathbf{v} = \lambda^3\mathbf{v},$$

so λ^3 is an eigenvalue of A^3 . Continuing this process — or for the aficionados amongst you, by using induction (try it!) — we see that

$$A^n\mathbf{v} = \lambda^n\mathbf{v}$$

for any positive integer n , so λ^n is an eigenvalue of A^n as claimed.

Exercise 2.

- (a) Since

$$\begin{pmatrix} 4 & 3 & 3 \\ -4 & -3 & -4 \\ 2 & 2 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ -8 \\ 4 \end{pmatrix} = 2 \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix},$$

the vector $(3, -4, 2)^T$ is an eigenvector of A corresponding to the eigenvalue 2.

(b) A somewhat tedious calculation shows that the characteristic polynomial p of A is

$$p(\lambda) = \det(A - \lambda I) = -\lambda^3 + 4\lambda^2 - 5\lambda + 2.$$

By (a), the polynomial p has a zero at 2, so using polynomial division we can factor p , giving

$$p(\lambda) = (2 - \lambda)(\lambda^2 - 2\lambda + 1) = (2 - \lambda)(\lambda - 1)^2.$$

Thus, the eigenvalues of A are 1 and 2. Using the usual method, we obtain the following bases for the eigenspaces:

$$N(A - I) = \text{Span}(\mathbf{v}_1, \mathbf{v}_2), \quad \text{where} \quad \mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},$$

$$N(A - 2I) = \text{Span}(\mathbf{v}_3), \quad \text{where} \quad \mathbf{v}_3 = \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix}.$$

(c) By (b) and the corollary to the Diagonalisation Theorem, the matrix A has 3 linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, so A is diagonalisable by the Diagonalisation Theorem. If we now define

$$P = (\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3) = \begin{pmatrix} -1 & -1 & 3 \\ 1 & 0 & -4 \\ 0 & 1 & 2 \end{pmatrix},$$

then P is invertible and

$$P^{-1}AP = D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

We can check whether we have made any mistake by calculating

$$AP = \begin{pmatrix} 4 & 3 & 3 \\ -4 & -3 & -4 \\ 2 & 2 & 3 \end{pmatrix} \begin{pmatrix} -1 & -1 & 3 \\ 1 & 0 & -4 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 6 \\ 1 & 0 & -8 \\ 0 & 1 & 4 \end{pmatrix},$$

$$PD = \begin{pmatrix} -1 & -1 & 3 \\ 1 & 0 & -4 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 6 \\ 1 & 0 & -8 \\ 0 & 1 & 4 \end{pmatrix},$$

so $AP = PD$ as required.

(d) Since $P^{-1}AP = D$ it follows that

$$A = PDP^{-1},$$

and hence that

$$A^5 = PD^5P^{-1}.$$

Thus

$$A^5 = \begin{pmatrix} -1 & -1 & 3 \\ 1 & 0 & -4 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1^5 & 0 & 0 \\ 0 & 1^5 & 0 \\ 0 & 0 & 2^5 \end{pmatrix} \begin{pmatrix} 4 & 5 & 4 \\ -2 & -2 & -1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 94 & 93 & 93 \\ -124 & -123 & -124 \\ 62 & 62 & 63 \end{pmatrix}.$$

Exercise 3.

(a) Let

$$A = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}.$$

Since

$$\det(A - \lambda) = \lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2),$$

the eigenvalues of A are -2 and 5 . The corresponding eigenspaces are

$$N(A + 2I) = \text{Span}(\mathbf{v}_1) \quad \text{where} \quad \mathbf{v}_1 = \begin{pmatrix} -3 \\ 4 \end{pmatrix},$$

$$N(A - 5I) = \text{Span}(\mathbf{v}_2) \quad \text{where} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Since the 2×2 matrix A has 2 linearly independent eigenvectors, A is diagonalisable and the diagonalising matrix is given by

$$P = \begin{pmatrix} -3 & 1 \\ 4 & 1 \end{pmatrix},$$

with

$$P^{-1}AP = D = \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}.$$

(b) Let

$$A = \begin{pmatrix} 6 & 4 & 2 \\ -7 & -6 & -5 \\ 4 & 4 & 4 \end{pmatrix}.$$

Since

$$\det(A - \lambda) = -\lambda^3 + 4\lambda^2 - 4\lambda = -\lambda(\lambda^2 - 4\lambda + 4) = -\lambda(\lambda - 2)^2,$$

the eigenvalues of A are 0 and 2. The corresponding eigenspaces are

$$N(A) = \text{Span}(\mathbf{v}_1) \quad \text{where} \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix},$$

$$N(A - 2I) = \text{Span}(\mathbf{v}_2) \quad \text{where} \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}.$$

Since the 3×3 matrix A has only 2 linearly independent eigenvectors A is not diagonalisable.

(c) Let

$$A = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Since A is triangular, its eigenvalues are 2 and 3. The corresponding eigenspaces are

$$N(A - 2I) = \text{Span}(\mathbf{v}_1, \mathbf{v}_2) \quad \text{where} \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$N(A - 3I) = \text{Span}(\mathbf{v}_3, \mathbf{v}_4) \quad \text{where} \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Since the 4×4 matrix A has 4 linearly independent eigenvectors, A is diagonalisable and the diagonalising matrix is given by

$$P = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with

$$P^{-1}AP = D = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

(d) Let

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Since A is triangular, its eigenvalues are 2 and 3. The corresponding eigenspaces are

$$N(A - 2I) = \text{Span}(\mathbf{v}_1, \mathbf{v}_2) \quad \text{where} \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$N(A - 3I) = \text{Span}(\mathbf{v}_3) \quad \text{where} \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Since the 4×4 matrix A has only 3 linearly independent eigenvectors, A is not diagonalisable.

Exercise 4. Let $A \in \mathbb{R}^{n \times n}$ be symmetric and suppose that $\mathbf{v}$ and $\mathbf{w}$ are eigenvectors corresponding to the distinct eigenvalues λ and μ , respectively, that is

$$\begin{aligned} A\mathbf{v} &= \lambda\mathbf{v}, \\ A\mathbf{w} &= \mu\mathbf{w}. \end{aligned}$$

Then by Lemma 7.27

$$\lambda(\mathbf{v} \cdot \mathbf{w}) = (A\mathbf{v}) \cdot \mathbf{w} = \mathbf{v} \cdot (A\mathbf{w}) = \mu(\mathbf{v} \cdot \mathbf{w}),$$

so

$$(\lambda - \mu)(\mathbf{v} \cdot \mathbf{w}) = 0.$$

But λ and μ are distinct, so $\lambda - \mu \neq 0$. Thus $\mathbf{v} \cdot \mathbf{w} = 0$, that is, $\mathbf{v}$ and $\mathbf{w}$ are orthogonal.

Exercise 5. Suppose that A is diagonalised by an orthogonal matrix Q , that is,

$$Q^T A Q = D$$

with D diagonal. Using the fact $Q^{-1} = Q^T$ (see Exercise 1 on Coursework 10), it follows that $Q^{-1} A Q = D$, so

$$A = Q D Q^{-1} = Q D Q^T.$$

Now D is diagonal, so $D^T = D$. Hence

$$A^T = (Q D Q^T)^T = (Q^T)^T D^T Q^T = Q D Q^T = A,$$

so A is symmetric as claimed.

As a consequence, the Spectral Theorem for Symmetric Matrices can also be formulated as follows: a matrix can be diagonalised by an orthogonal matrix if and only if A is symmetric.

Exercise 6.

(a) Let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Since

$$\det(A - \lambda) = \lambda^2 - 2\lambda - 3 = (\lambda + 1)(\lambda - 3),$$

the eigenvalues of A are -1 and 3 . The corresponding eigenspaces are

$$N(A + I) = \text{Span}(\mathbf{v}_1) \quad \text{where} \quad \mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

$$N(A - 3I) = \text{Span}(\mathbf{v}_2) \quad \text{where} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Normalising the two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ produces

$$\mathbf{u}_1 = \frac{1}{\|\mathbf{v}_1\|} \mathbf{v}_1 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \quad \mathbf{u}_2 = \frac{1}{\|\mathbf{v}_2\|} \mathbf{v}_2 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix},$$

so if we let

$$Q = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

Then

$$Q^T A Q = D = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}.$$

We can check whether we have made a mistake by calculating

$$Q^T Q = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

so Q is indeed orthogonal as claimed. Moreover

$$Q^T A Q = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} = D,$$

as required.

(b) Let

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}.$$

Since

$$\det(A - \lambda) = \lambda^2 - 7\lambda + 6 = (\lambda - 1)(\lambda - 6),$$

the eigenvalues of A are 1 and 6 . The corresponding eigenspaces are

$$N(A - I) = \text{Span}(\mathbf{v}_1) \quad \text{where} \quad \mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

$$N(A - 6I) = \text{Span}(\mathbf{v}_2) \quad \text{where} \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

Normalising the two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ yields the two vectors

$$\mathbf{u}_1 = \frac{1}{\|\mathbf{v}_1\|} \mathbf{v}_1 = \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}, \quad \mathbf{u}_2 = \frac{1}{\|\mathbf{v}_2\|} \mathbf{v}_2 = \begin{pmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix},$$

so if we let

$$Q = \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix},$$

then

$$Q^T A Q = D = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}.$$

Again, as in (a) you are strongly encouraged to check that Q is orthogonal and does the job it is supposed to do.

(c) Let

$$A = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}.$$

The characteristic polynomial p is given by

$$p(\lambda) = \det(A - \lambda I) = -\lambda^3 + 12\lambda + 16.$$

Since we are told that -2 is an eigenvalue of A we can factor p using polynomial division giving

$$p(\lambda) = -(\lambda + 2)(\lambda^2 - 2\lambda - 8) = -(\lambda + 2)(\lambda - 4)(\lambda + 2) = -(\lambda + 2)^2(\lambda - 4),$$

so the eigenvalues of A are -2 and 4 . The corresponding eigenspaces are

$$N(A + 2I) = \text{Span}(\mathbf{x}_1, \mathbf{x}_2) \quad \text{where} \quad \mathbf{x}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},$$

$$N(A - 4I) = \text{Span}(\mathbf{x}_3) \quad \text{where} \quad \mathbf{x}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

The eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are not orthogonal, so we need to orthogonalise them using the Gram Schmidt process:

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}.$$

Normalising the orthogonal eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{x}_3$ yields the orthonormal eigenvectors

$$\mathbf{u}_1 = \frac{1}{\|\mathbf{v}_1\|} \mathbf{v}_1 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \quad \mathbf{u}_2 = \frac{1}{\|\mathbf{v}_2\|} \mathbf{v}_2 = \begin{pmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix}, \quad \mathbf{u}_3 = \frac{1}{\|\mathbf{x}_3\|} \mathbf{x}_3 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix},$$

so if we let

$$Q = \begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix},$$

then

$$Q^T A Q = D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

Again, as in (a) you are strongly encouraged to check that Q is orthogonal and does the job it is supposed to do.